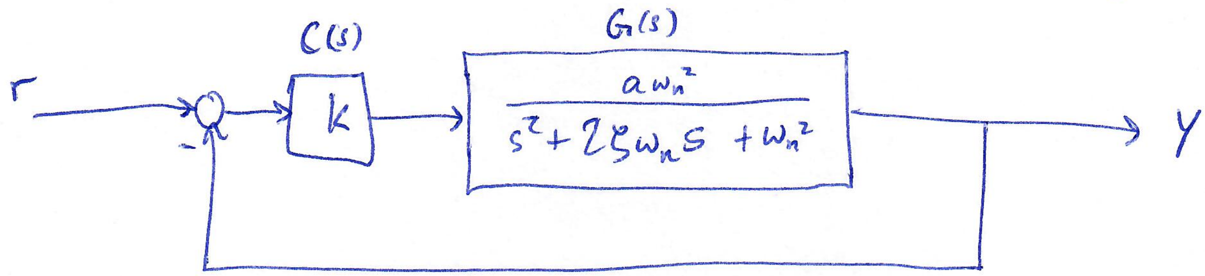
ME 4555 - Lecture 22 - PI control and FVT

We discussed last time how feedback control can be used to improve the response of a 1^{st} order system. Let's examine higher order systems now. Consider proportional control for a 2^{nd} order sys:



Closed-loop transfer function:

$$\frac{Y}{R} = \frac{CG}{1+CG} = \frac{\dfrac{ak\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}}{1+\dfrac{ak\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}} = \frac{ak\omega_n^2}{s^2+2\zeta\omega_n s+(1+ak)\omega_n^2}$$

old parameter		new parameter		
gain	a	gain	$\frac{ak}{ak+1}$	← same as 1^{st} order system. Always less than 1.
natural frequency	ω_n	natural frequency	$\sqrt{1+ak}\,\omega_n$	← faster as we increase ω_n. (higher frequency)
damping ratio	ζ	damping ratio	$\frac{1}{\sqrt{1+ak}}\zeta$	← smaller as we increase k. (less damping)
poles	$-\zeta\omega_n \pm i\omega_n\sqrt{1-\zeta^2}$	poles	$-\zeta\omega_n \pm i\omega_n\sqrt{1+ak-\zeta^2}$	← poles have same real part (same settling time t_s) larger imaginary part.

How can we find the steady-state value $\lim_{t\to\infty} y(t)$ without having to calculate the response and take the limit? We can use the Final Value Theorem (FVT).

> If $\lim_{t\to\infty} y(t)$ exists and the transfer function $Y(s)$ has all poles in the left-half plane (stable) plus at most one pole at the origin, then:
>
> $$\lim_{t\to\infty} y(t) = \lim_{s\to 0} s\,Y(s)$$

A rough proof:

$$\lim_{t\to\infty} y(t) = \Big[y(t) \Big]_{t=0}^{t=\infty} + y(0)$$

$$= \int_0^\infty \dot{y}(t)\,dt + y(0)$$

$$= \left(\lim_{s\to 0} \int_0^\infty e^{-st}\,\dot{y}(t)\,dt \right) + y(0)$$

$$= \lim_{s\to 0} \mathcal{L}\{\dot{y}(t)\}(s) + y(0)$$

$$= \lim_{s\to 0} \left(s\,Y(s) - \cancel{y(0)} \right) + \cancel{y(0)}$$

$$= \lim_{s\to 0} s\,Y(s)$$

these steps only work if the assumptions of the theorem are satisfied.

Lets apply the FVT to some familiar examples.

To find steady-state response to a step input,

$$Y(s) = \underbrace{\tfrac{1}{s}}_{\text{step input}} \underbrace{G(s)}_{\text{transfer function}}. \qquad \text{So, } \lim_{t\to\infty} y(t) = \lim_{s\to 0} sY(s) = \lim_{s\to 0} s \cdot \tfrac{1}{s} G(s) = \boxed{G(0).}$$

1st order system: $G(s) = \dfrac{K}{\tau s + 1}$. $\lim_{t\to\infty} y(t) = G(0) = K$.

2nd order system: $G(s) = \dfrac{K\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2}$. $\lim_{t\to\infty} y(t) = G(0) = K$.

1st order with proportional feedback: $G(s) = \dfrac{ak}{\tau s + 1 + ak}$. $\lim_{t\to\infty} y(t) = G(0) = \dfrac{ak}{1+ak}$.

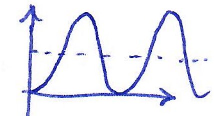
2nd order with proportional feedback: $G(s) = \dfrac{ak\omega_n^2}{s^2 + 2\xi\omega_n s + (1+ak)\omega_n^2}$. $\lim_{t\to\infty} y(t) = G(0) = \dfrac{ak}{1+ak}$.

This explains why we can't get $y(t) \to 1$ using proportional feedback! If $G(0)$ is constant, then $\lim_{t\to\infty} y(t) = \dfrac{KG(0)}{1+KG(0)} < 1$.

<u>Beware</u> if FVT assumptions are violated, we get nonsense results.

<u>ex 1</u>: $\dfrac{1}{s^2+1}$. $G(0) = 1$.

But this is undamped.
Poles at $\pm j$ (<u>not</u> in left-half plane)
so does <u>not</u> satisfy FVT assumptions



Limit does not exist!

<u>ex 2</u>: $\dfrac{1}{s-1}$. $G(0) = -1$.

This is unstable! Pole at $s=1$
(not in left-half plane)

Limit is $+\infty$!

We can still achieve a limit of 1 in $\frac{C(s)G(s)}{1+C(s)G(s)}$ if we make $C(s)$ so that $\lim_{s\to 0} C(s) = \infty$, then the limit will be 1! So pick $\boxed{C(s) = \underbrace{k_p}_{\text{proportional term}} + \underbrace{\frac{k_i}{s}}_{\text{integral term.}}}$

This is called a proportional-integral (PI) controller.

<u>ex</u> 1^{st} order system: $G(s) = \frac{1}{\tau s+1}$

$$\frac{C(s)G(s)}{1+C(s)G(s)} = \frac{(k_p + k_i/s)\left(\frac{1}{\tau s+1}\right)}{1+(k_p+k_i/s)\frac{1}{(\tau s+1)}} = \boxed{\frac{k_p s + k_i}{\tau s^2 + (k_p+1)s + k_i}}$$

It has become a 2^{nd} order system! Note that when $s=0$, we get 1, so the step-response has a steady-state value of 1 as long as the system is stable (FVT).

Now that we have a second order system, there may be overshoot. Also, this is not a standard 2^{nd} order system because it has an "s" in the numerator, so the response $y(t)$ may not look exactly the same as expected... But the exponential envelope and damped frequency ω_d are the same (only depend on the pole locations). [see step response simulation!]